

5.1] $X(t)$ has $R_X(f) = \begin{cases} R_0 & |f| \leq B \\ 0 & |f| > B \end{cases}$

$$Y(t) = \frac{dX(t)}{dt}$$

a) Power spectral density of $Y(t)$

$$R_Y(f) = |H(f)|^2 R_X(f) \rightarrow Y(f) = H(f) X(f) = \int \left(\frac{d}{dt} \right)^n x(t) \xrightarrow{F}$$

$$(j2\pi f)^n X(f) \int = Y(f) = (j2\pi f) X(f) \Rightarrow |H(f)|^2 = 4\pi^2 f^2$$

$$R_Y(f) = 4\pi^2 f^2 R_X(f)$$

$$R_Y(f) = \begin{cases} 4\pi^2 f^2 R_0 & |f| \leq B \\ 0 & f > B \end{cases}$$

b) Ratio between output power $Y(f)$ $|f| \leq B/2$ and the total output power

$$\int_{-B/2}^{B/2} 4\pi^2 f^2 R_0 df = 4\pi^2 R_0 \left(\frac{f^3}{3} \right)_{-B/2}^{B/2} = 4\pi^2 R_0 \left(\frac{(B/2)^3}{3} + \frac{(B/2)^3}{3} \right)$$

$$= 8\pi^2 R_0 \left(\frac{B^3}{8 \cdot 3} \right) = \pi^2 R_0 \cdot \frac{B^3}{3}$$

Total power:

$$\int_{-B}^B 4\pi^2 f^2 R_0 df = 4\pi^2 R_0 \left(\frac{f^3}{3} \right)_{-B}^B = 4\pi^2 R_0 \left(2 \frac{(B)^3}{3} \right) =$$

$$= 8\pi^2 R_0 \frac{B^3}{3}$$

$$\text{Ratio} = \frac{\pi^2 R_0 \cdot \frac{B^3}{3}}{8\pi^2 R_0 \cdot \frac{B^3}{3}} = \frac{1}{8}$$

5.16

$X(t) = S_1(t) + N_1(t)$ connected to the input of a lowpass filter

$S_1(t) = A \cos(\omega_0 t + \Phi)$ A and Φ are independent R.V.

$N_1(t)$: white noise with spectral density R_0

$$P_A = E\{A^2\}$$

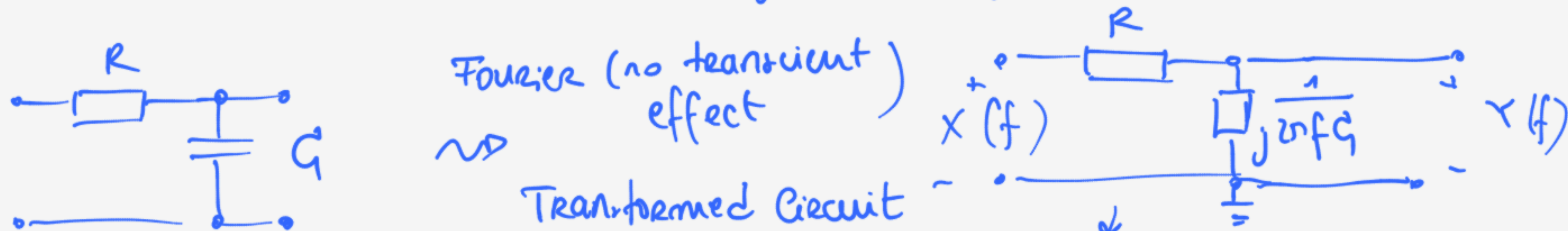
Φ uniform $[0, 2\pi)$

$$Y(t) = S_2(t) + N_2(t) \quad \text{SNR} = \frac{E\{S_2^2(t)\}}{E\{N_2^2(t)\}}$$

determine $T = RC$ that maximizes SNR.

C and R are linear elements (ideally) hence:

$$Y(t) = h(t) * X(t) \quad \text{and} \quad Y(f) = H(f) X(f)$$



$$(X(f) - Y(f)) \frac{1}{j2\pi f C} = R Y(f) \quad \Leftrightarrow \quad \frac{X(f) - Y(f)}{R} = \frac{Y(f)}{1/j2\pi f C}$$

$$\frac{X(f)}{j2\pi f C} = Y(f) \left(R + \frac{1}{j2\pi f C} \right)$$

$$X(f) \frac{1}{j2\pi f C} = Y(f) \frac{R + \frac{1}{j2\pi f C}}{1}$$

$$X(f) \frac{1}{1 + j2\pi f C R} = Y(f)$$

$$H(f) = \frac{1}{1 + j2\pi f T}$$

Two options to calculate power $\rightarrow r_y(\tau)$ for $\tau=0$ convolution exponential!
 $\rightarrow \int R_y(f) df$ ✓

$$R_y(f) = R_x(f) |H(f)|^2 \quad |H(f)|^2 = \frac{1}{1 + (2\pi f RC)^2}$$

$$r_x(\tau) = E[(A \cos(\omega_0(t+\tau) + \Phi) + N_1(t+\tau))(A \cos(\omega_0 t + \Phi) + N_1(t))] =$$

= {indep A and Φ } =

$$+ E[N_1(t+2) A \cos(\omega_0 t + \phi)] + E[N_1(t) A \cos(\omega_0 (t+2) + \phi)]$$

N_1 indep. from rest and ϕ average

$$+ E[N_1(t+2) N_1(t)] = \frac{E[A^2]}{2} \cos(\omega_0 \tau) + \underbrace{R_{\phi}(\tau)}_{\text{white noise}}$$

tutorial 4

Taking the power transform we will get $R_x(f)$:

$$R_x(f) = \frac{P_A}{4} (\delta(f-f_0) + \delta(f+f_0)) + \underbrace{R_0}_{\text{noise}} ; \text{ where } f_0 = \frac{\omega_0}{2\pi}$$

$$R_y(f) = R_x(f) |H(f)|^2 \quad \text{signal of interest.}$$

$$R_y(f) = \frac{P_A}{4} (\delta(f-f_0) + \delta(f+f_0)) \left(\frac{1}{1 + (2\pi f RC)^2} \right) + \frac{R_0}{1 + (2\pi f RC)^2}$$

$$P_{S_2} = \int \frac{P_A}{4} (\delta(f-f_0) + \delta(f+f_0)) \frac{1}{1 + (2\pi f RC)^2} df =$$

$$= \frac{P_A}{4} \left(\frac{1}{1 + (2\pi f_0 RC)^2} + \frac{1}{1 + (2\pi f_0 RC)^2} \right) = \frac{P_A}{2} \frac{1}{1 + (2\pi f_0 RC)^2}$$

$$P_{N_2} = \int_{-\infty}^{\infty} \frac{R_0}{1 + (2\pi f RC)^2} df = \frac{R_0}{(2\pi RC)^2} \int_{-\infty}^{\infty} \frac{1}{\frac{1}{(2\pi RC)^2} + f^2} df = \left\{ \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctg\left(\frac{x}{a}\right) \right\}$$

$$= \frac{R_0}{(2\pi RC)^2} \cdot (2\pi RC) \cdot \arctg(2\pi RC f) \Big|_{-\infty}^{\infty} =$$

$$= \frac{R_0}{2\pi RC} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{R_0}{2RC}$$



$$\text{Hence the SNR} = \frac{P_A / (1 + (\omega_0 RC)^2)}{R_0 / RC} = \frac{P_A \cdot RC}{R_0 (1 + (\omega_0 RC)^2)}$$

we now want to maximize the SNR, as a function of $T = RC$

$$\max \left\{ \frac{P_A T}{R_0 (1 + (\omega_0 T)^2)} \right\} = \max \left\{ \frac{T}{1 + (\omega_0 T)^2} \right\}$$

→ Take first derivative

$$\frac{1 \cdot (1 + (\omega_0 T)^2) - 2T\omega_0^2 \cdot T}{(1 + (\omega_0 T)^2)^2} =$$

$$= \frac{1 - \omega_0^2 T^2}{(1 + (\omega_0 T)^2)^2} = 0$$

$$1 = \omega_0^2 T^2$$

$$T^2 = 1/\omega_0^2$$

$T = \pm 1/\omega_0$ we do not want negative resistor or capacitor:

$$T = 1/\omega_0 \rightarrow \underline{\underline{RC = 1/\omega_0}}$$

• For the problem to be strictly correct one would have to check if this is actually a maximum by seeing that the second order derivative for $T = 1/\omega_0$ is negative. We will however trust the headline in this case.

6.2. $Y(n) = 0.5Y(n-1) + X(n)$

$X(n)$ has 0 mean

Mean power of $Y(n)$

ACF $r_X(k) = \delta(k)$

$E[Y(n+1)Y(n)]$

$Y(n)$ is an order 1 A-R process.

⇒ $Y(n-1)$ and $X(n)$ are independent since $Y(n-1)$ is "made of" previous samples of X and $X(n)$ is white.

$$E[Y(n)^2] = E[(0.5Y(n-1) + X(n))(0.5Y(n-1) + X(n))] =$$

$$= (0.5)^2 E[Y(n-1)^2] + E[X(n)^2] = (0.5)^2 r_Y(0) + r_X(0) =$$

$$r_Y(0) = (0.5)^2 r_Y(0) + r_X(0)$$

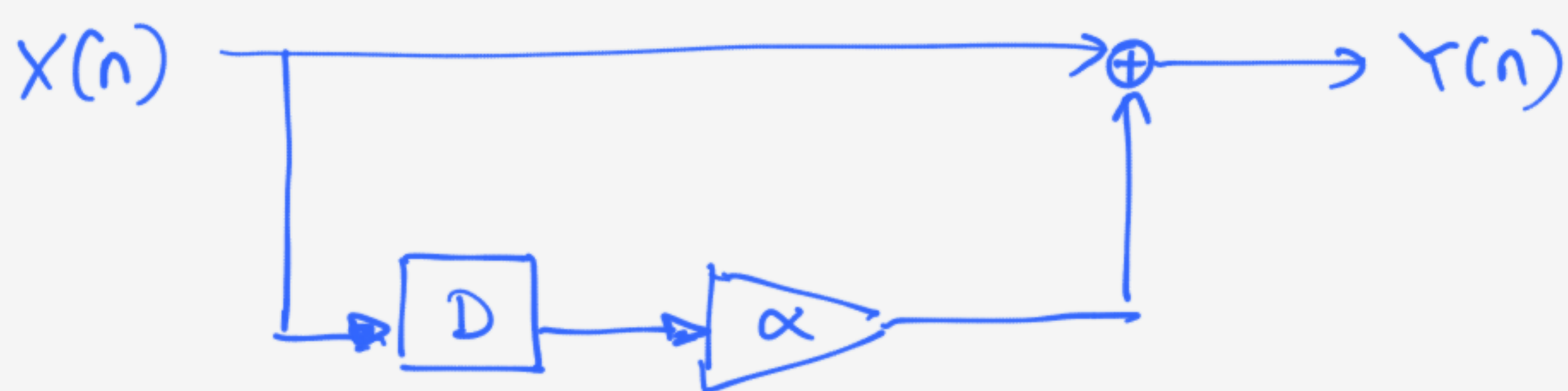
$$(1 - \frac{1}{4}) r_Y(0) = r_X(0)$$

$$r_Y(0) = \frac{4}{3} r_X(0) \quad \underline{\underline{r_Y(0) = 4/3}}$$

$$\begin{aligned}
 \bullet E[Y(n+1)Y(n)] &= E[(0.5Y(n) + X(n+1))(Y(n))] = \\
 &= E[0.5Y^2(n)] + E[X(n+1)Y(n)] = \\
 &\quad \text{indep and } X(n+1) \text{ has } 0 \text{ mean} \\
 &= 0.5r_x(0)
 \end{aligned}$$

$$r_x(1) = 0.5r_x(0) = 0.5 \cdot \frac{4}{3} = \frac{2}{3}$$

6.11] WSS $X(n)$ ACF $r_x(k)$



α that achieves the least mean output power as possible.

$$Y(n) = \alpha X(n-1) + X(n)$$

$$\begin{aligned}
 E[Y^2(n)] &= E[(X(n) + \alpha X(n-1))(X(n) + \alpha X(n-1))] = \\
 &= r_x(0) + 2\alpha r_x(1) + \alpha^2 r_x(0) = (1 + \alpha^2)r_x(0) + 2\alpha r_x(1)
 \end{aligned}$$

we want to minimize, we take derivative.

$$2\alpha r_x(0) + 2r_x(1) = 0 \quad \alpha = \frac{-r_x(1)}{r_x(0)}$$

$$\alpha = \frac{-r_x(1)}{r_x(0)} \quad ; \quad 2^{\text{nd}} \text{ order derivative } 2r_x(0) > 0 \Rightarrow \text{it's a minimum}$$

$$\Rightarrow \text{Determine ratio } \frac{E[Y(n)^2]}{E[X(n)^2]} \bigg|_{\alpha = \frac{-r_x(1)}{r_x(0)}}$$

$$\begin{aligned}
 E[Y(n)^2] &= \left(1 + \frac{r_x^2(1)}{r_x^2(0)}\right) r_x(0) - 2 \frac{r_x(1)}{r_x(0)} r_x(1) = \\
 &= r_x(0) + \frac{r_x^2(1)}{r_x(0)} - \frac{2r_x^2(1)}{r_x(0)} = r_x(0) - \frac{r_x^2(1)}{r_x(0)}
 \end{aligned}$$

$$\text{Ratio} = \frac{E[Y^2(n)]}{r_x(0)} = \frac{r_x(0) - \frac{r_x^2(1)}{r_x(0)}}{r_x(0)} = 1 - \left(\frac{r_x(1)}{r_x(0)} \right)^2$$